Dynamic Monopolies and Vaccination

Lucia Penso

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Joint with Bessy, Dourado, Ehard, Rautenbach

Dynamic Monopolies

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Informal Definition

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Dynamic monopolies are a simple graph-theoretical model for various types of viral processes in networks.

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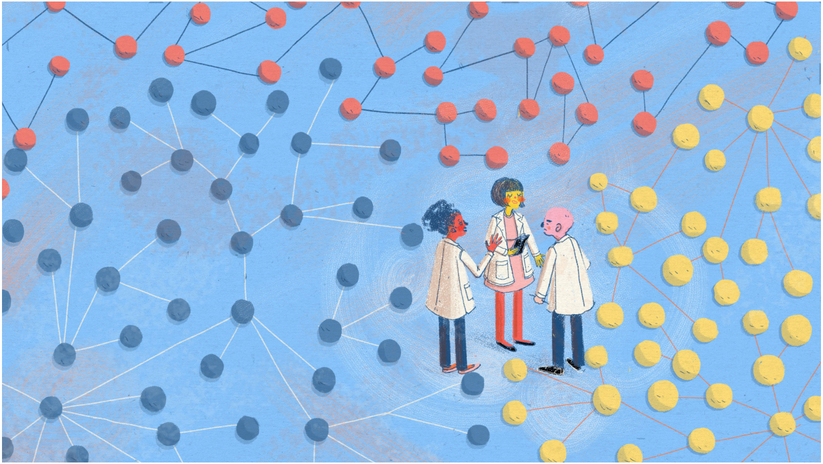
...examples for things that can spread...

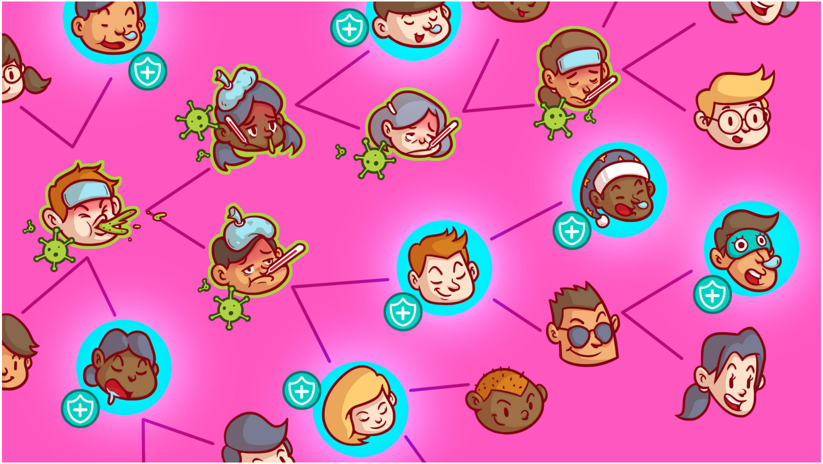
- opinions,
- computer viruses,
- diseases,
- products,
- habits,
- ...

Dynamic Monopolies



(picture taken from www.quantamagazine.org)

Dynamic Monopolies



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Dynamic Monopolies

Let G be a graph.

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return $H_{(G,\tau)}(D)$;

Dynamic Monopolies

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$$\underbrace{D \subseteq V(G) : H_{(G,\tau)}(D) = V(G)}_{D \text{ is a dynamic monopoly of } (G,\tau)}$$

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$$\min\left\{|D| : \underbrace{D \subseteq V(G) : H_{(G,\tau)}(D) = V(G)}_{D \text{ is a dynamic monopoly of } (G,\tau)}\right\}$$

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-

$$D \text{ is a a dynamic monopoly of } (G,\tau)$$
$$\Updownarrow$$
$$V(G) \setminus D \text{ is a } (d_G - \tau)\text{-degenerate set in } G.$$

Dynamic Monopolies

Theorem (Chen '09, P et al. '11)

Determining $\mathrm{dyn}(G, 2)$ *is NP-hard.*

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Determining $\mathrm{dyn}(G, 2)$ *is NP-hard.*

...even hard to approximate.

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A simple reduction algorithm for trees...

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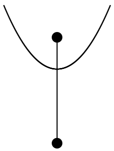

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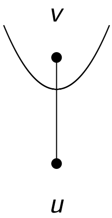

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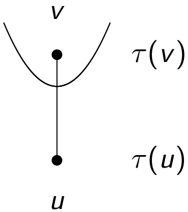

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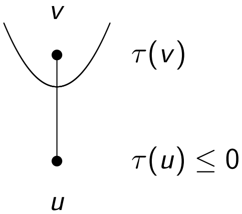

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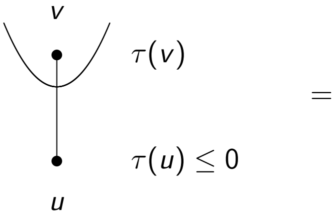

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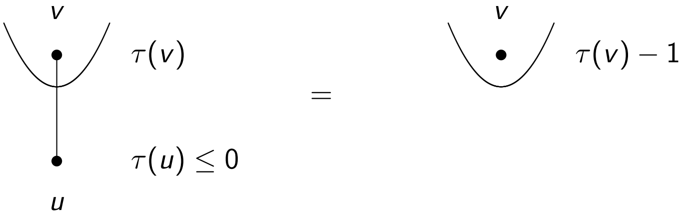

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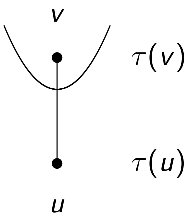

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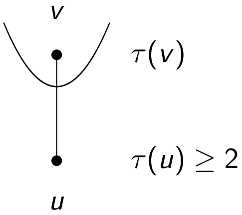

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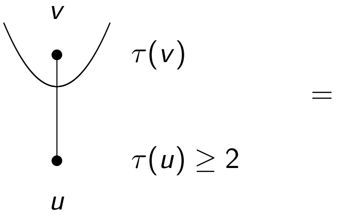

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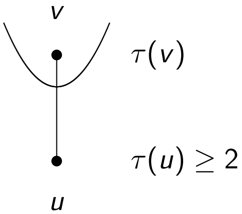

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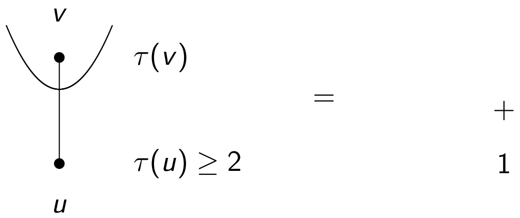

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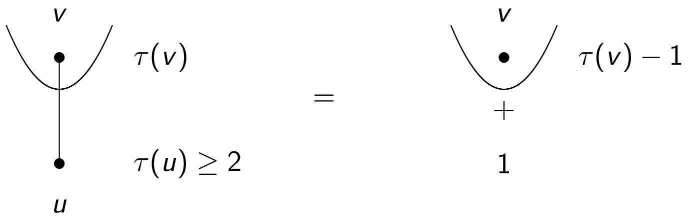

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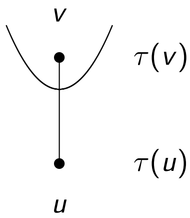

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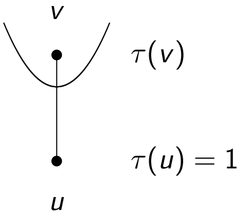

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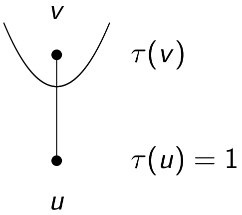

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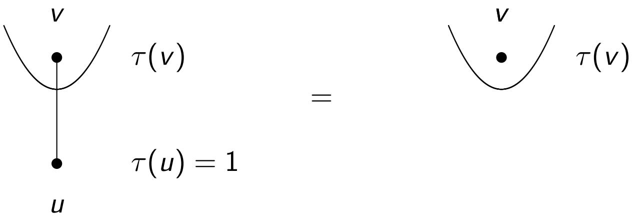

Dynamic Monopolies

Theorem (Chen '09, P et al. '11)

For a given pair (T, τ), where T is a tree, $\mathrm{dyn}(T, \tau)$ can be determined in linear time.

Dynamic Monopolies

Two extensions of this result:

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Theorem (P et al. '11)

For a given pair (G, τ)*, where* G *has blocks of bounded order,* $\mathrm{dyn}(G, \tau)$ *can be determined in polynomial time.*

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Theorem (P et al. '11)

For a given pair (G, τ), where G has blocks of bounded order, $\mathrm{dyn}(G, \tau)$ can be determined in polynomial time.

Theorem (Ben-Zwi et al. '11)

For a given pair (G, τ), where G has order n and treewidth w, $\mathrm{dyn}(G, \tau)$ can be determined in $n^{O(w)}$ time.

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The last result suggests that $\mathrm{dyn}(G, \tau)$ might only be tractable for tree-structured graphs.

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Lemma (P et al. '11)

If (G, τ) is such that

- *G is a 2-connected chordal graph and*
- *$\tau \leq 2$,*

then $\{u, v\}$ is a dynamic monopoly for (G, τ) for every edge uv of G.

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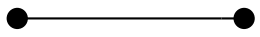

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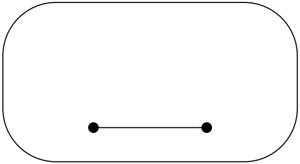

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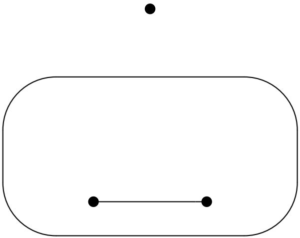

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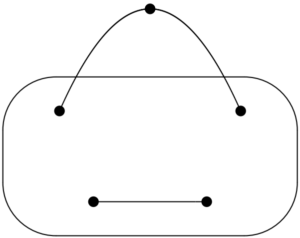

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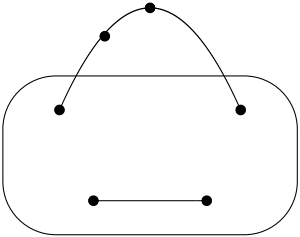

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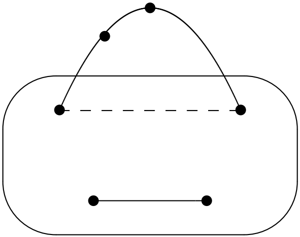

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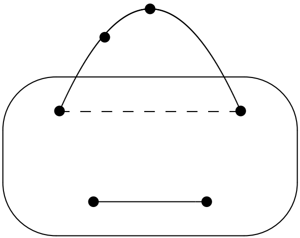

$\square$

Dynamic Monopolies

Theorem (P et al. '11)

For a given pair (G, τ)*, where*

- G *is chordal and*
- $\tau \leq 2$,

$\mathrm{dyn}(G, \tau)$ *can be determined in polynomial time.*

Dynamic Monopolies

Lemma (Chiang et al. '13)

Let t be a non-negative integer.
If (G, τ) is such that

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- *$\tau \leq t$,*

then C is a dynamic monopoly for (G, τ) for every clique C of order t.

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In particular,

$$\mathrm{dyn}(G, \tau) \leq t.$$

Dynamic Monopolies

Problem

Is there a polynomial time algorithm that determines

$$\mathrm{dyn}(G, \tau)$$

for a given pair (G, τ) *such that*

- G *is chordal, and*
- τ *is bounded?*

Dynamic Monopolies

Theorem (BEPR '18)

Let t be a non-negative integer.
For a given pair (G, τ), where

- *G is an interval graph, and*
- *$\tau \leq t$,*

$\mathrm{dyn}(G, \tau)$ can be determined in polynomial time.

Dynamic Monopolies

Theorem (BEPR '18)

For a given triple (G, τ, k)*, where*

- G *is a chordal graph,*
- τ *is a threshold function for* G*, and*
- k *is a positive integer,*

it is NP-complete to decide whether $\mathrm{dyn}(G, \tau) \leq k$*.*

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(jump a little?!)

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Let G be an interval graph of order n, and let $\tau \leq t$ be a threshold function.

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Let G be an interval graph of order n, and let $\tau \leq t$ be a threshold function. Let $(I(u))_{u \in V(G)}$ be an interval representation using closed intervals with distinct endpoints $x_1 < x_2 < \ldots < x_{2n}$.

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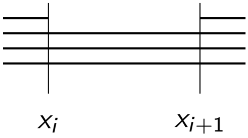

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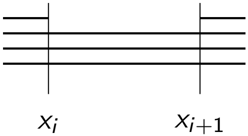

Every minimal vertex cut of G is a clique of the form

$$C_i = \Big\{ u \in V(G) : [x_i, x_{i+1}] \subseteq I(u) \Big\}.$$

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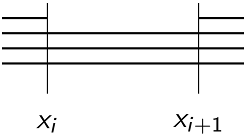


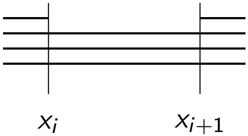
Every minimal vertex cut of G is a clique of the form

$$C_i = \Big\{ u \in V(G) : [x_i, x_{i+1}] \subseteq I(u) \Big\}.$$

For $c_i = |C_i|$, we have $|c_i - c_{i+1}| = 1$.

Dynamic Monopolies

Let G be an interval graph of order n, and let $\tau \leq t$ be a threshold function. Let $(I(u))_{u \in V(G)}$ be an interval representation using closed intervals with distinct endpoints $x_1 < x_2 < \ldots < x_{2n}$.



Every minimal vertex cut of G is a clique of the form

$$C_i = \Big\{ u \in V(G) : [x_i, x_{i+1}] \subseteq I(u) \Big\}.$$

For $c_i = |C_i|$, we have $|c_i - c_{i+1}| = 1$.

Let $j_1 < j_2 < \ldots < j_{k-1}$ be the indices i with

$$c_i < \min \Big\{ c_{i-1}, c_{i+1}, t \Big\}$$

and let $j_k = 2n - 1$.

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Let $V_i = C_1 \cup \cdots \cup C_{j_i}$, $G_i = G[V_i]$, and $B_i = C_{j_i}$.

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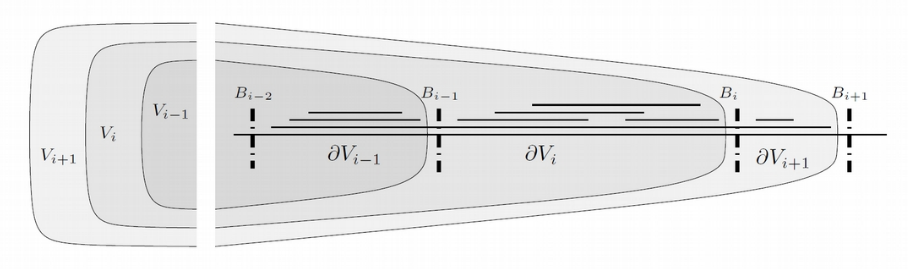

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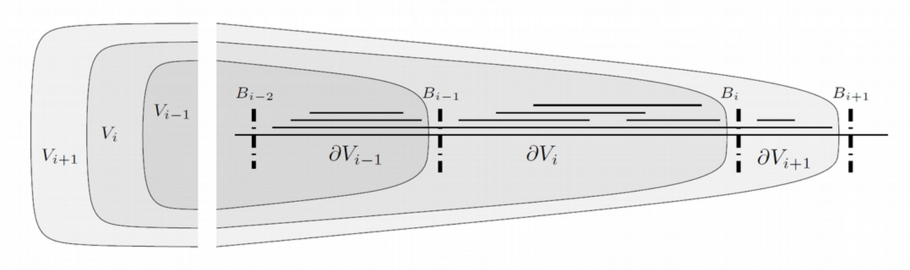

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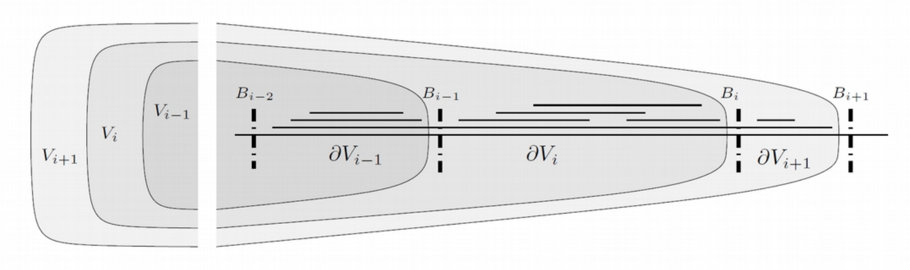

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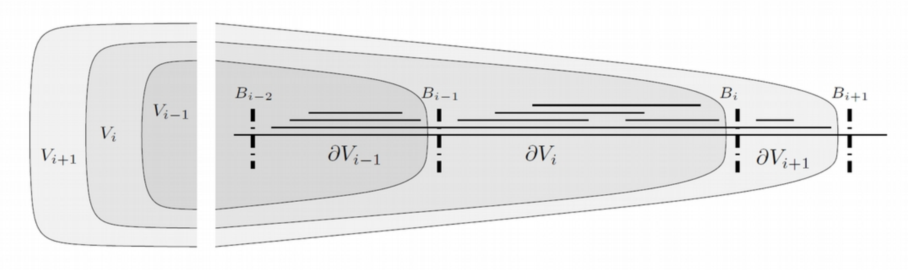

$|B_i| < t$. No vertex in $V_i \setminus B_i$ has a neighbor in $V(G) \setminus V_i$.

Let $\partial V_i = (V_i \setminus V_{i-1}) \cup B_{i-1}$, and $\partial G_i = G[\partial V_i]$.

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Each ∂G_i is either a clique of order at most t or t-connected.

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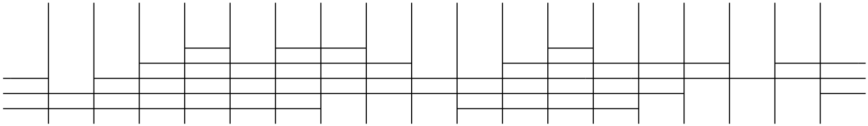

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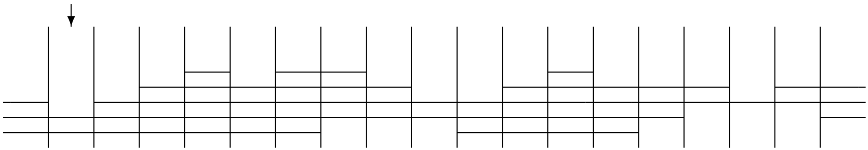

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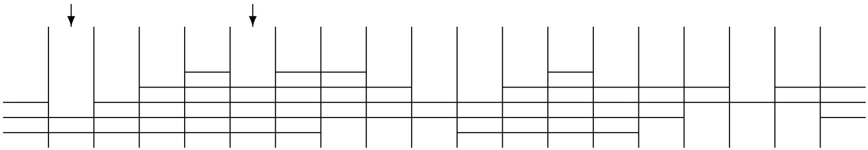

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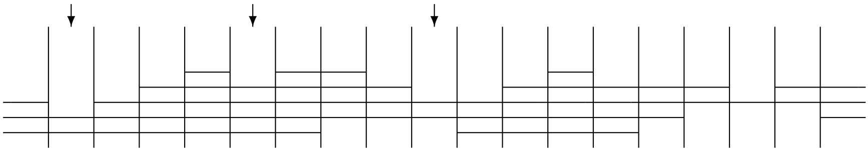

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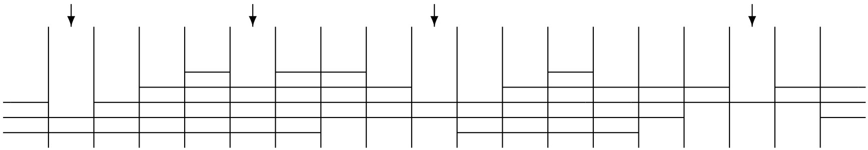

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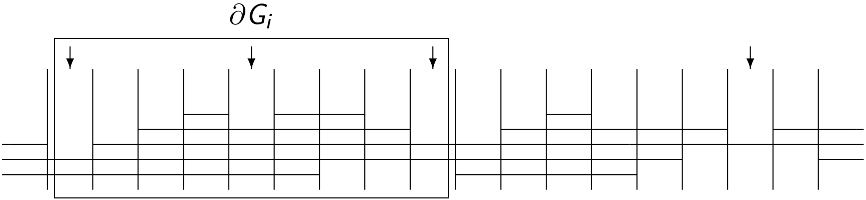

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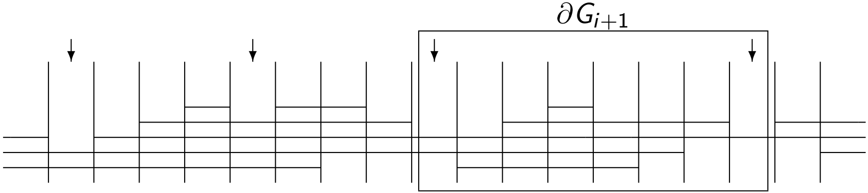

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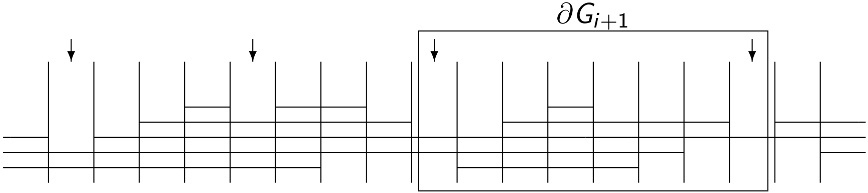

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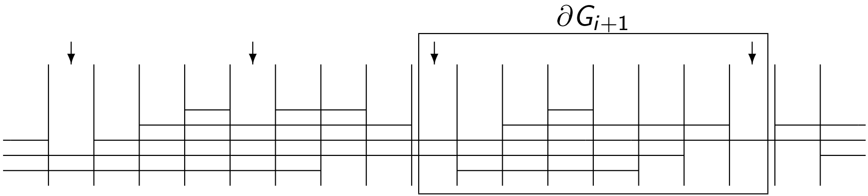

□

(jump a little?!)

Dynamic Monopolies

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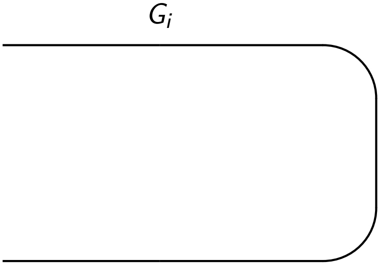

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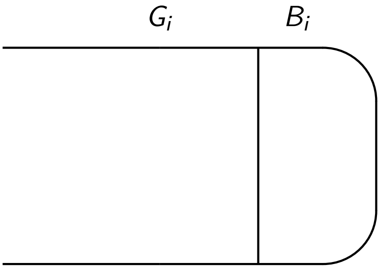

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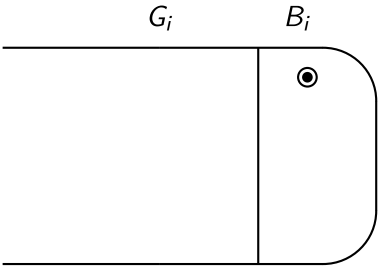

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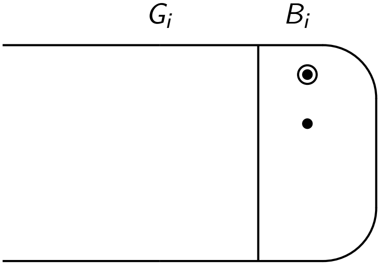

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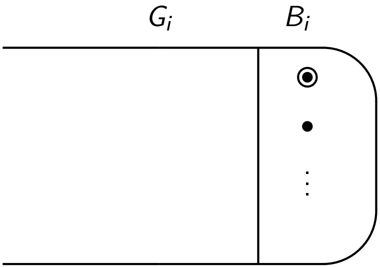

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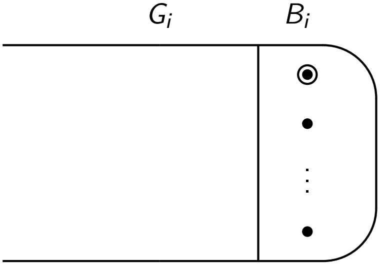

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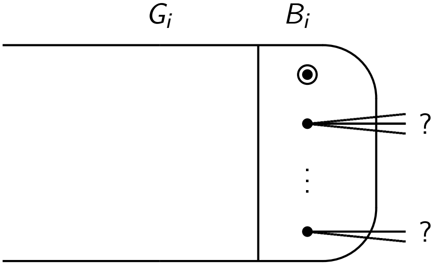

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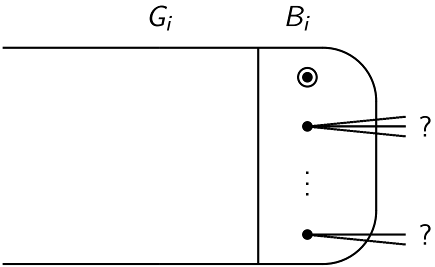

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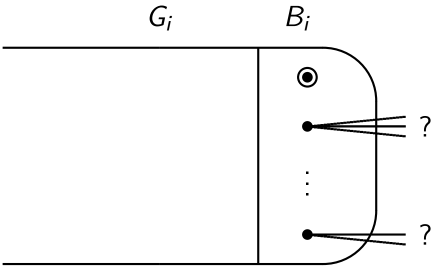

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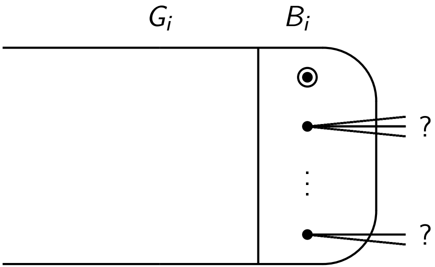

A local cascade for G_i is a triple $(X_i, \prec_i, \rho_i)$, where

1. (i) X_i is a subset of B_i,
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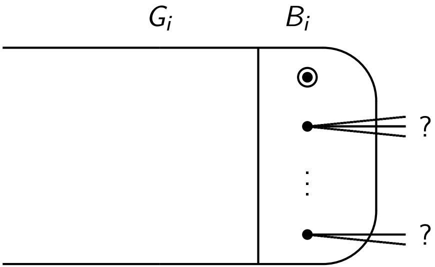

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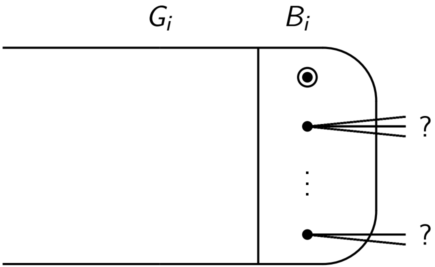

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There are $O\Big(2^{t-1}(t-1)!(n+1)^{t-1}\Big)$ such local cascades.

Dynamic Monopolies

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For a local cascade $(X_i, \prec_i, \rho_i)$ for G_i, let

$$\mathrm{dyn}_i(X_i, \prec_i, \rho_i)$$

be the minimum order of a subset Y_i of $V_i \setminus B_i$ such that the following conditions hold:

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- (c) or $u_j \in B_i \setminus X_i$ and $\left|N_G(u_j) \cap \{u_1, \ldots, u_{j-1}\}\right| \geq \tau(u_j) - \rho_i(u_j)$.

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□

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If G *is a graph with vertex degrees* $d_G(u_1) \leq \ldots \leq d_G(u_{n(G)})$, *then the maximum of* $\mathrm{dyn}(G, \tau)$ *over all non-negative choices for* τ *such that the average threshold is at most* $\bar{\tau}$ *equals*

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Requiring $0 \leq \tau \leq d_G$ in the above setting, the problem becomes NP-hard for planar graphs but can be solved efficiently for trees.

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\mathrm{vacc}_2(G, \tau, b) &= \max \left\{ \mathrm{dyn}(G - Y, \tau) : Y \in \binom{V(G)}{b} \right\}
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\mathrm{vacc}_2(G,\tau,b) &= \max\left\{\mathrm{dyn}(G-Y,\tau) : Y \in \binom{V(G)}{b}\right\}\\
\mathrm{vacc}_3(G,\tau,\iota_{\max},b) &= \max\Big\{\mathrm{dyn}(G,\tau+\iota) : \iota \in \mathbb{Z}^{V(G)},\\
&\qquad 0 \le \iota \le \iota_{\max},\ \text{and } \iota(V(G)) = b\Big\}
\end{aligned}$$

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- $\mathrm{vacc}_2(T, \tau, b)$ *can be determined in* $O\left(n^3(b+1)^2\right)$ *time.*

Thank you for the attention!